Iinoya, K., and R. J. Altpeter, "Inverse Response in Process Control," *Ind. Eng. Chem.*, **54**, 39 (1962).
Kloubec, R. E., "Direct Digital Control of Systems Subject to Stochastic Disturbances and Control Effort Constraints," Ph.D. thesis, Purdue Univ., Lafayette, Ind. (1971).
Kuo, B. C., Analysis and Synthesis of Sampled-Data Control Systems, pp. 394, Prentice-Hall, Englewood Cliffs, N. J. (1963).
Lees, F. P., "Step Responses of Outlet Liquid Concentration in a Simple Gas Absorption Column," *Ind. Eng. Chem. Fundamentals*, **9**, 512 (1970).
Lim, H. C., and S. G. Bankoff, "Wiener-Hopf Methods for Unstable, Nonminimum Phase Processes: Part I. Optimal Regulators," *AIChE J.*, **16**, 233 (1970).
Lim, H. C., and S. G. Bankoff, "Wiener-Hopf Methods for Unstable, Nonminimum Phase Processes. Part II. Application to a Stirred Tank Reactor," *AIChE J.*, **16**, 240 (1970).
Tou, J. T., and K. S. P. Kumar, "Statistical Design of Discrete-Data Control Systems Subject to Power Limitation," *J. Franklin Inst.*, **272**, 171 (1961).
Wood, R. M., "The Frequency Response of Multicomponent Distillation Columns," *Trans. Inst. Chem. Eng.*, **45**, T190 (1967).

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Effect of Axial Dispersion on Microbial Growth

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The extent of microbial growth in a continuous flow reactor is influenced by various degrees of fluid mixing which may range from one extreme of plug-flow to the other extreme of completely mixed flow. To assess the effect of mixing on the extent of microbial growth various flow models such as the tanks-in-series model and the axial dispersion model are often employed. Such flow models have been successfully applied to biological waste treatment processes by Murphy (1971) and Lee et al. (1971).

The purpose of this note is to present and discuss the analytical solutions of differential equations describing the extent of microbial growth based on the axial dispersion model under the assumption that the specific growth rate does not depend on the substrate concentration. The assumption permits the differential equations to be solved analytically and provides useful limiting case for more complicated substrate limited systems. These results should be useful in connection with fermentation processes and biological waste water treatment.

Specifically two problems are considered. The first is concerned with steady state solution with nonsterile feed. The second is concerned with the transient and steady state solution with sterile feed.

STEADY STATE GROWTH WITH NONSTERILE FEED [$X(0) > 0$]

The differential equation based on the axial dispersion model for this case can be represented in the following dimensionless form with Danckwerts' boundary conditions:

$$\frac{1}{Pe}\frac{d^2X}{dz^2} - \frac{dX}{dz} + KX = 0 \tag{1}$$

$$X - \frac{1}{Pe}\frac{dX}{dz} = 1 \qquad \text{at} \quad z = 0 \tag{2}$$

$$\frac{dX}{dz} = 0 \qquad \text{at} \quad z = 1 \tag{3}$$

where Pe is the Peclet number. Pe is a measure of the degree of mixing, ranging from $Pe = 0$ (completed mixed flow) to $Pe = \infty$ (plug flow).

The solution can be obtained by a standard method such as Laplace transformation for the following three cases:

(i) $Pe > 4K$

$$X(z) = \frac{2\exp\left(\frac{Pe\,z}{2}\right)\left[\sinh\frac{aPe}{2}(1-z) + a\cosh\frac{aPe}{2}(1-z)\right]}{(1+a^2)\sinh\frac{aPe}{2} + 2a\cosh\frac{aPe}{2}} \tag{4}$$

where

$$a \equiv \sqrt{1 - \frac{4K}{Pe}}$$

(ii) $Pe = 4K$

$$X(z) = \frac{1 + \frac{Pe}{2}(1-z)}{1 + \frac{Pe}{4}}\exp\left(\frac{Pe\,z}{2}\right) \tag{5}$$

(iii) $Pe < 4K$

$$X(z) = \frac{2\exp\left(\frac{Pe\,z}{2}\right)\left[\sin\frac{bPe}{2}(1-z) + b\cos\frac{bPe}{2}(1-z)\right]}{(1-b^2)\sin\frac{bPe}{2} + 2b\cos\frac{bPe}{2}} \tag{6}$$

where

$$b \equiv \sqrt{\frac{4K}{Pe} - 1}$$

Equation (4) may be compared to the solution for first-order kinetics by Danckwerts (1953) and Wehner and Wilhelm (1956). However, K is positive in the present system because the microbial growth is an autocatalytic kinetic process. Thus additional solutions of Equations (5) and (6) must be obtained to describe the mixing conditions when $Pe \leq 4K$. It should be mentioned that Stevens (1966) obtained a solution equivalent to Equation (4) at the reactor exit but failed to obtain solution for the case $Pe < 4K$. Hence, he concluded that a flow reactor cannot be represented by the dispersion model for $Pe < 4K$. It is shown here that his conclusion is not true.

Furthermore, one must make sure that the solutions obtained in all cases satisfy the physical constraint of positive cell concentration in $0 \leq z \leq 1$. It is quite clear that Equations (4) and (5) meet this requirement because they are hyperbolic functions. For the case $Pe < 4K$, one observes that the numerator in Equation (6) has a positive value at $z = 1$ and must remain positive for the entire range $0 \leq z \leq 1$. This implies that the denominator must also be positive in order that $X(z) > 0$. Therefore the solution of Equation (6) is meaningful only when the following relation also holds true.

$$\tan\frac{bPe}{4} \leq \frac{1}{b} \tag{7}$$

or

$$K \leq \frac{\alpha_1^2}{Pe/4} + Pe/4 \tag{8}$$

where α_1 is the smallest positive root of the following equation (Abramowitz, 1965)

$$\alpha \tan\alpha = \frac{Pe}{4} \tag{9}$$

Thus, Equation (6) is valid when Pe and K are related such that

$$\frac{Pe}{4} < K \leq \frac{Pe}{4} + \frac{\alpha_1^2}{Pe/4} \tag{10}$$

The above discussion can be summarized in Figure 1. If the pair of K and Pe lies in region (I) there is no solution to the problem. Physically this means that the microbial cells grow faster than can be carried away by the exit flow stream. In regions (II) and (III) the finite solutions exist as presented by Equations (6) and (4) respectively.

Figure 2 shows the effect of axial dispersion on the exit cell concentration. This figure may be used in the design of flow reactors for the production of microbial cells.

TRANSIENT AND STEADY STATE GROWTH CHARACTERISTICS FOR STERILE FEED [X (0) = 0]

In general the cells are seldom fed continuously to the fermenters. The start-up of a continuous flow fermenter usually takes the following steps: (1) the cells are inoculated to the fermenter containing the suitable medium; (2) the fermenter is operated initially batchwise and switched to continuous run after the exponential growth phase is reached; and (3) the medium and air are continuously fed to the fermenter but not the cell (that is, sterilized feed). It is important to see under what limiting operating conditions the cells can sustain their growth in the fermenter without being washed out.

It is well known that in a CSTR the washout of microbial cells occurs when the dilution rate is greater than the maximum specific growth rate while in a plug-flow fermenter washout always occurs at any flow rate if the

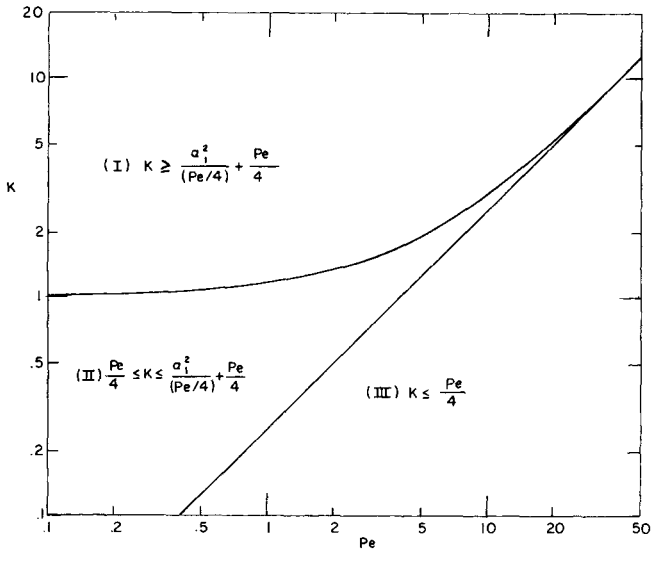

Fig. 1. The relation between K and Pe for the solution to Equations (1) to (3).

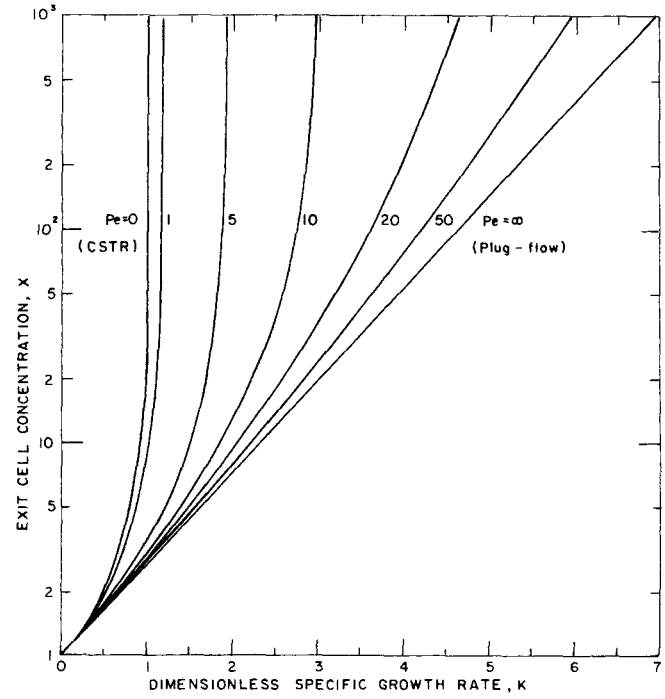

Fig. 2. Effect of axial dispersion on exit cell concentrations for nonsterile feed.

cells are not fed continuously. Between these two extreme cases the washout phenomenon clearly depends on the degree of mixing and the holding time. To determine the mixing effect on washout let us consider the start-up problem employing the dispersion model expressed by the following partial differential equations and boundary conditions.

$$\frac{\partial X}{\partial \theta} = \frac{1}{Pe}\frac{\partial^2 X}{\partial x^2} - \frac{\partial X}{\partial z} + KX \qquad (11)$$

$$X = 1 \quad \text{for all } z \quad \text{at} \quad \theta = 0 \qquad (12)$$

$$X - \frac{1}{Pe}\frac{\partial X}{\partial z} = 0 \quad \text{at} \quad z = 0 \quad \text{for} \quad \theta > 0 \qquad (13)$$

$$\frac{\partial X}{\partial z} = 0 \quad \text{at} \quad z = 1 \quad \text{for} \quad \theta \geq 0 \qquad (14)$$

where X is the dimensionless cell concentration based on the initial concentration [Equation (12)] at the moment of switching from batch to continuous operation.

The solution to this problem may be obtained through the use of variable-separation technique followed by the Laplace transformation and is as follows:

$$X(\theta, z) = \exp\left(\frac{Pe\, z}{2}\right) \sum_{n=1}^{\infty} A_n \exp(-\lambda_n \theta) \left[\frac{Pe}{4\alpha_n}\sin(2\alpha_n z) + \cos(2\alpha_n z)\right] \qquad (15)$$

where α_n, $n = 1, 2, \ldots$ are real roots of

$$\alpha \tan \alpha = \frac{Pe}{4} \quad \text{or} \quad \alpha \cot \alpha = -\frac{Pe}{4} \qquad (16)$$

and

$$\lambda_n = \frac{Pe}{4} + \frac{\alpha_n^2}{(Pe/4)} - K \qquad (17)$$

$$A_n = \frac{8\frac{\alpha^2_n}{Pe}\left[\left(\frac{Pe}{4}\right)^2 + \alpha_n^2\right]}{\frac{Pe}{4} + \left(\frac{Pe}{4}\right)^2 + \alpha_n^2} \qquad (18)$$

Several interesting points may be noted.

1. If all the eigenvalues λ_n, $n = 1, 2, \ldots$ are positive for given Pe and K then washout will occur as $\theta \to \infty$. From Equation (17) we can see that the washout condition is

$$K < \frac{\alpha_1^2}{\left(\frac{Pe}{4}\right)} + \frac{Pe}{4} \qquad (19)$$

where α_1 is the smallest positive root satisfying Equation (16).

This condition coincides with the regions (II) and (III) in Figure 1.

In these regions steady state solutions do not exist for the present problem (sterile feed) but do exist for the previous problem (nonsterile feed). Washout condition of Equation (19) is consistent with the results obtained by Fan et al. (1970) for more complicated kinetics under the assumption that micromixing is in the state of maximum mixedness.

2. If one or more of λ_n are negative, then the cells will grow without bound as $\theta \to \infty$. Of course in actual fermentations cell growth will eventually be limited by the depletion of substrate. This implies that if the pair of K and Pe is in region (I) cell growth is self-sustained.

It is possible to use these results for the substrate limited case if some maximum cell concentration is considered. For example, in the problem of nonsterile feed, let S_0 be the inlet substrate concentration then the maximum cell concentration is limited to $X_{max} = 1 + \Delta X_{max}$ where $\Delta X_{max} = Y_d S_0$, where Y_d is the yield factor.

3. If the smallest λ_n is zero then the steady state solution is possible. However this is an unstable steady state solution because any flow fluctuation will make the smallest λ nonzero and the cells will either grow without bound or be washed out.

CONCLUDING REMARKS

Under the assumption of constant specific growth rate analytical solutions for the extent of microbial growth based on the axial dispersion model have been obtained. Conditions for the existence of steady state solutions for sterile and nonsterile feeds are shown to depend on the degree of fluid mixing and specific growth rate.

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NOTATION

A_n = defined by Equation (18)
a = defined by Equation (4)
b = defined by Equation (6)
D = axial dispersion coefficient
K = dimensionless specific growth rate, $\mu L/u$
L = total reactor length
Pe = dimensionless dispersion number, uL/D
S_0 = inlet substrate concentration
t = real time
u = fluid velocity
x = reactor length
X = dimensionless cell concentration
Y_d = yield factor
z = dimensionless reactor length, x/L
α_n = real roots of Equation (16)
λ_n = defined by Equation (17)
θ = dimensionless time, tu/L
μ = specific cell growth rate

LITERATURE CITED

Abramowitz, M., and I. A. Stegun, (eds.), Handbook of Mathematical Functions, p. 224, Dover, New York (1965).

Danckwerts, P. V., "Continuous Flow Systems," *Chem. Eng. Sci.*, **2**, 1 (1953).

Fan, L. T., L. E. Erickson, P. S. Shah, and B. I. Tsai, "Effect of Mixing on the Washout and Steady State Performance of Continuous Cultures," *Biotech Bioeng.*, **12**, 6, 10, 19 (1970).

Lee, S. S., L. E. Erickson, and L. T. Fan, "Modeling and Optimization of a Tower—Type Activated Sludge System," in *Biotech Bioeng. Symp.*, R. C. Canale, (ed.), 2, 141 (1971).

Murphy, K. L., "The Significance of Flow Patterns and Mixing in Biological Waste Treatment," in *ibid.*, **2**, 35 (1971).

Stevens, N. J., "Mixing Effects in the Continuous Growth of Microorganisms," *Chem. Eng. Progr. Symp. Ser., No. 69*, **62**, 50 (1966).

Wehner, J. F., and R. H. Wilhelm, "Boundary Conditions of Flow Reactor," *Chem. Eng. Sci.*, **6**, 89 (1956).

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